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THE ROYAL SOCIETY

Market liquidity and its effect on option valuation and hedging

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The model presented in this paper attempts to quantify the concept of liquidity and establishes a relation between various measures of market performance. Informational inefficiency is argued to be the main reason for the unavailability of an asset at its equilibrium price. The asset, however, can always be purchased at a higher price or sold at a lower price, depending on the market expectations, unless trading has ceased. The mathematical model describing the asset-price behaviour together with arbitrage considerations enables us to estimate the component of the bid–ask spread resulting from imperfect information. The impact of the market liquidity on hedging an option with another option as well as the underlying asset itself is also examined. In this last case uncertainty cannot be completely eliminated from the hedged portfolio, although a unique risk-minimizing strategy is found.

Keywords: option pricing; bid–ask spread; transaction costs; market liquidity; asymptotic methods

1. Introduction

Usually liquidity is defined as the ability to transact immediately and with negligibly small impact on the price of a security regardless of the size of the transaction. For a market to be liquid, trading should be both informationally and transactionally efficient, i.e. prices should fully reflect all available information and transactions should be performed at prices that differ insignificantly from equilibrium prices. Consequently, a bid–ask spread will be minimal in this case. Absence of liquidity makes itself manifest significantly during market crashes; for instance, Wang *et al.* (1990) finds that the illiquidity of the S&P 500 index, as measured by the bid–ask spread, increased up to eight times during the market crash of 1987. Although the liquidity of a market depends strongly on its structure, market makers are found to be the major providers of liquidity in many futures markets, setting up the highest bid and the lowest offer, and it is reasonable to assume that this is also the case for other markets as well.

Informational and transactional efficiencies are not independent; the former drives the latter, characterized by the realized bid–ask spread, which contains information about liquidity rather than being the measure of liquidity itself, as asserted in Wang

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et al. (1990). It is widely held that a bid–ask spread can be split into two parts. The first part, due to inventory carrying costs, is of little interest in this work. It is analysed in Amihud & Mendelson (1980) and Ho & Stoll (1981). Stochastic costs of carry are considered in a number of recent papers, for example, Gibson & Schwartz (1990), Brennan (1991) and Schwartz (1997).

The second part of the spread, that resulting from outstanding information†, is examined by Copeland & Galai (1983) and Glosten & Milgrom (1985). In this work we consider this second component but approach it in a continuous-time setting under the assumption that the return on a stock over a short period of time is normally distributed. Also, a concept of memory is introduced into the stock-price process. This makes it possible to link the concepts of informational inefficiency, unavailability of the underlying and market trends together and model their impact on the price of a security. This method has similarities with the works on stochastic convenience yield mentioned above. The assumption that the market price does not reflect all available information implies that a Markovian process is not suitable to model stock-price behaviour.

As mentioned earlier, one of the distinguishing features inherent in illiquid markets is a frequent inability to buy or sell an asset at its equilibrium price. The reason for this is that not all the information available about the asset is reflected fully in its current price‡ and hence the asset behaviour becomes locally predictable, i.e. an excess demand will result in the increase of the stock price over the next time-step and likewise an excess supply will result in a decrease of the price. As a result, if the return on a stock is exceeding or going to exceed (with certainty) the risk-free interest rate, the stock is unlikely to be available for purchase at its intrinsic price, S (its equilibrium price without outstanding information). Similarly, a falling market leads to an inability to sell the stock for S. This results naturally in different sale and purchase prices quoted by a market maker, which are adjusted in accordance with the current market trend, as well as his expectations, in order to eliminate any potential arbitrage opportunities.

The liquidity service of a market maker allows one to trade continuously in time knowing his bid and ask prices before the transaction is commenced. Because of the inertia associated with continuous trading there will always be an uncompensated supply or demand which is not reflected completely in the current stock price and this leads to market trends. However, trends cannot stay for a significant length of time before prices move to eliminate them.

This article is organized as follows. In § 2 we introduce a model to describe a stock-price behaviour which takes into account the stock liquidity. The option valuation model is developed in § 3 and optimal bounds for the bid–ask spread are determined. A scheme for finding the underlying parameters of the model is suggested in § 4. Finally, in § 5 we consider hedging an option with the underlying asset. The hedging scheme developed is important for tailor-made derivatives contracts when no other claims on the asset are traded in the market. In the appendix we consider briefly the related but simpler problem of hedging an option when the underlying asset is unavailable for technical reasons (that is, there are randomly distributed times during which the asset is not available for *any* price).

† That is, not all information available is represented by the current stock price.
‡ The case when unavailability of the asset is caused by technical reasons is considered in Appendix A.

2. The stock-price model

Following the previous discussion, one plausible model for the process followed by the intrinsic stock price, S, is

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + m\,\mathrm{d}t, \tag{2.1}$$

where the trend m is a normally distributed zero-mean process and r is the risk-free rate.

The assumption that m_t is a delta-correlated process, i.e. $E(m_t m_\tau) = \delta(t-\tau)$, leads to a geometrical Brownian motion; trends do not exist in this case, they are arbitraged immediately. Assume now that m_t is driven by the following system of the stochastic differential equations:

$$\mathrm{d}m = \ell\,\mathrm{d}t, \qquad \mathrm{d}\ell = -\omega^2 m\,\mathrm{d}t - \gamma\ell\,\mathrm{d}t + \sigma\omega^2\,\mathrm{d}\beta_t, \tag{2.2}$$

where β_t is a unit-variance Brownian-motion process which reflects trades and arrival of new information. The process followed by ℓ represents our expectations about the stock's future behaviour (since $\ell = \mathrm{d}^2(\log S)/\mathrm{d}t^2$ is analogous to an acceleration). The coefficient $\gamma > 0$ represents damping and restricts information propagation between trades. Clearly, very old stock history is irrelevant for the current stock price; large γ corresponds to short memory and small γ corresponds to long memory. The restoring term ω ensures that trends will not persist over large time-scales (greater than $O(1/\omega)$), i.e. the market moves back to equilibrium; large ω leads to a quick restoring action. The parameter σ plays a role analogous to the volatility in a usual geometric Brownian motion, see (5.4). The homogeneous system (2.2) is asymptotically stable if $\gamma > 0$, since the eigenvalues of the homogeneous problem have negative real parts. Although the Brownian-motion process β_t does not enter equation (2.1) directly, the stock price S responds immediately to the arrival of new information. This is because the system (2.1), (2.2) can be written as

$$\mathrm{d}x = Ax\,\mathrm{d}t + B\,\mathrm{d}\beta_t,$$

where the pair (A, B) is absolutely controllable (as shown below) and

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -\gamma \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ \sigma\omega^2 \end{pmatrix}, \quad x = \begin{pmatrix} \log(S) - rt \\ m \\ \ell \end{pmatrix}.$$

The rank of the Kalman matrices $[B, AB, A^2B]$ is three, and this means that $Q(t) = \mathrm{cov}(x_t)$ is a non-singular matrix for all $t > 0$. Hence, $x(t)$ has the density function

$$p(x,t) = \frac{\exp(-x'Q(t)^{-1}x)}{(2\pi\det(Q(t)))^{3/2}}$$

and therefore, at any future time there is a positive probability that x will reach any given neighbourhood of positive measure in $\mathbb{R}^3$ (see Davis (1977) for details).

For simplicity, we consider now the case where $\gamma \to \infty$, $\omega \to \infty$ and $\omega^2/\gamma \to \rho < \infty$. We return to the full model only in the conclusion. This yields the following model for a stock-price behaviour:

$$\mathrm{d}S = (r+m)S\,\mathrm{d}t, \tag{2.3}$$

$$\mathrm{d}m = -\rho m\,\mathrm{d}t + \sigma\rho\,\mathrm{d}\beta_t. \tag{2.4}$$

In this situation the parameter ρ represents both restoring and damping terms. To understand the financial implications, the system (2.3), (2.4) is represented as

$$\frac{1}{\rho}\ddot{y} = -(\dot{y} - r) + \sigma w_t,$$

where $y = \log(S)$ and $w_t = \mathrm{d}\beta_t/\mathrm{d}t$ is a white-noise process, the weak derivative of the Brownian motion. If $\dot{y} - r > 0$ at some stage then it will remain so for a sufficiently small time-interval because of the continuity of $\dot{y}$, which follows from (2.4). As a result of this, a damping force is applied to the growth rate of the share. Financially it means that the situation when the rate of return on the share exceeds its equilibrium rate cannot exist for longer than $O(1/\rho)$ before prices move to eliminate it. A similar case occurs when $\dot{y} - r < 0$. The term $m = \dot{y} - r$ can be thought of as a market trend and it follows an Orstein–Uhlenbeck process. The limit $\rho \to \infty$ gives the geometrical Brownian motion process,

$$\mathrm{d}S = (r + \tfrac{1}{2}\sigma^2)S\,\mathrm{d}t + \sigma S\,\mathrm{d}\beta_t.$$

This fact will also be verified in § 5.

To investigate the properties of the process followed by m_t consider the solution of (2.4),

$$m_t = m_0 \mathrm{e}^{-\rho t} + \sigma\rho \int_0^t \mathrm{e}^{-\rho(t-s)} w_s\,\mathrm{d}s.$$

Here, in order to avoid having to consider the entire history of trading, m_0 is set to be a normally distributed random variable (independent of the white-noise process, w_t), and with zero mean and variance $\frac{1}{2}\sigma^2\rho$ (as with this particular choice of the initial condition the transition process is absent). Hence the expected value and the correlation function of m_t are

$$\mathcal{E}(m_t) = 0, \qquad k(t,\tau) = \mathcal{E}(m_t m_\tau) = \tfrac{1}{2}\sigma^2\rho\mathrm{e}^{-\rho|\tau - t|}.$$

We employ these formulae in § 4 to estimate the underlying parameters of this model.

The process followed by the stock price, S, is not Markovian since it remembers the past through m_t, whereas the process followed by the state vector $Z_t = (S_t, m_t)'$ is Markovian. Thus the price of a derivative security should depend on Z_t rather than S_t alone. In other words, the price of a derivative depends to some extent on the history of the stock-price behaviour through the market trend.

3. Option pricing

In this section we consider effects of (2.3), (2.4) on option pricing. The stock price, S, is not the price of a traded security in the usual sense. We assume, however, that the derivatives market (particularly the futures market) is more liquid, and there are securities written on this stock which are tradeable in the Black–Scholes sense. Let $V(t, S, m)$ be the price of such a security, a continuously differentiable function in t and having continuous second partial derivatives with respect to S, m. Then, from Itô's lemma, the stochastic differential $\mathrm{d}V$ is

$$\mathrm{d}V = \frac{\partial V}{\partial t}\,\mathrm{d}t + \frac{\partial V}{\partial S}\,\mathrm{d}S + \frac{\partial V}{\partial m}\,\mathrm{d}m + \tfrac{1}{2}\sigma^2\rho^2\frac{\partial^2 V}{\partial m^2}\,\mathrm{d}t.$$

All tradeable securities in the market should have the same market price of risk, $\lambda = \lambda(t, S, m)$. Hence the option pricing equation is

$$\frac{\partial V}{\partial t} + (r+m)S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2\rho^2\frac{\partial V^2}{\partial m^2} - (\sigma\rho\lambda + \rho m)\frac{\partial V}{\partial m} - rV = 0. \tag{3.1}$$

Further, for simplicity of calculations λ is assumed to be a constant, which will be found later in this section.

The substitutions $S = E\mathrm{e}^x$, $t = T - \tau$, $V = Eg(x, \tau)$, where E is the exercise price of the option and T is its expiry date, give the equation

$$\frac{\partial g}{\partial \tau} + (r+m)\frac{\partial g}{\partial x} + \tfrac{1}{2}\sigma^2\rho^2\frac{\partial^2 g}{\partial m^2} - (\sigma\rho\lambda + \rho m)\frac{\partial g}{\partial m} - rg = 0. \tag{3.2}$$

For the initial condition we take $g(x, m, 0) = \delta(x)$. The general problem can be solved by convolution of Green's function g with the actual pay-off.

On applying the Fourier transform

$$u(\xi, \eta, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, m, \tau)\mathrm{e}^{-\mathrm{i}(\xi x + \eta m)}\,\mathrm{d}x\,\mathrm{d}m,$$

to equation (3.2) one obtains the first-order hyperbolic initial-value problem

$$\begin{gathered}\frac{\partial u}{\partial \tau} + (\xi - \rho\eta)\frac{\partial u}{\partial \eta} = (\mathrm{i}\xi r - \mathrm{i}\sigma\rho\lambda\eta - \tfrac{1}{2}\sigma^2\rho^2\eta^2 + \rho - r)u, \\ u(\xi, \eta, 0) = 2\pi\delta(\eta).\end{gathered} \tag{3.3}$$

The characteristics of this equation are given by the following system:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \xi - \rho\eta,$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = u(\mathrm{i}\xi(\tau)r - \mathrm{i}\sigma\rho\lambda\eta(\tau) - \tfrac{1}{2}\sigma^2\rho^2\eta^2(\tau) + \rho - r), \quad u(0) = 2\pi\delta(\eta(0)).$$

Inverting the Fourier transform shows that Green's function is

$$g(x, m, \tau) = \frac{\mathrm{e}^{-r\tau}}{\sqrt{2\pi\sigma^2 c(\tau)}}\exp\left(-\frac{(r\tau - \sigma\lambda(\tau - p(\tau)) + x + mp(\tau))^2}{2\sigma^2 c(\tau)}\right), \tag{3.4}$$

where

$$c(\tau) = \tau - \frac{2}{\rho}(1 - \mathrm{e}^{-\rho\tau}) + \frac{1}{2\rho}(1 - \mathrm{e}^{-2\rho\tau}), \qquad p(\tau) = \frac{1}{\rho}(1 - \mathrm{e}^{-\rho\tau}). \tag{3.5}$$

Schwartz (1997) asserts that a closed-form solution to an equation of the type (3.2) was first obtained by Jamshidian & Fein (1990). The formula for pricing a European call option is found as the convolution of the Green's function (3.4) with the pay-off $\max(\mathrm{e}^x - 1, 0)$ and, in financial variables, is

$$C(S, m, t) = S\mathrm{e}^{mp(T-t)+\alpha(T-t)}N(\tilde{d}_1) - E\mathrm{e}^{-r\,(T-t)}N(\tilde{d}_2),$$

where†

$$\tilde{d}_1 = \frac{\log(S/E) + mp(T-t) + \alpha(T-t) + r\,(T-t) + \frac{1}{2}\sigma^2 c(T-t)}{\sigma c^{1/2}(T-t)},$$
$$\tilde{d}_2 = \frac{\log(S/E) + mp(T-t) + \alpha(T-t) + r(T-t) - \frac{1}{2}\sigma^2 c(T-t)}{\sigma c^{1/2}(T-t)},$$
$$\alpha(T-t) = \lambda\sigma(p(T-t) - (T-t)) + \tfrac{1}{2}\sigma^2 c(T-t),$$

and $N(x)$ is the cumulative probability distribution function for a standardized normal variable:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,\mathrm{d}y.$$

It is easy to verify that the price of the forward contract with the delivery price K is

$$f(t) = S\mathrm{e}^{mp(T-t)+\alpha(T-t)} - K\mathrm{e}^{-r(T-t)}.$$

The forward price F is defined as the delivery price which would make the contract have zero value. Therefore,

$$F = S\mathrm{e}^{mp+\alpha(T-t)+r(T-t)}. \tag{3.6}$$

Not only is the intrinsic stock price, S, reflected by the forward price F but so is the current market trend m. The study of Varson & Selby (1994) reveals the relationship between the price of a stock and the price implied by a call option on this stock. According to them the implied stock prices lead the observed stock prices by about 15 minutes on average (see their paper for details). This lead/lag relationship can be seen from the formula for the forward price F, which takes into account the market trend m, unlike the intrinsic price S. Usually derivatives markets are more liquid than stock markets, which accounts for this sort of cross-market inefficiency.

As has been mentioned earlier, stock markets, where an underlying is not always available either for sale or purchase because of excess supply or demand, have the common feature of locally predictable behaviour. This does not give rise, however, to arbitrage opportunities. An excess growth rate (exceeding the risk-free rate of interest) naturally results in the increase of the purchase price, eliminating any profitable opportunities. Analogously, a deficient growth rate leads to the decrease of the current sale price. Now the estimates for these two prices will be found in order to eliminate arbitrage opportunities, which are inconsistent with any market equilibrium theory, both in the stock and derivatives markets. We find that the least purchase, S_{p}, and the biggest sale price, S_{s}, which can be quoted by a market maker, are given by

$$S_{\mathrm{p}}(t) = \sup_{T\geqslant t} \mathrm{e}^{-r(T-t)}F(t), \qquad S_{\mathrm{s}}(t) = \inf_{T\geqslant t} \mathrm{e}^{-r(T-t)}F(t).$$

Indeed, if we had $S_{\mathrm{p}}(t) < \sup_{T\geqslant t} \mathrm{e}^{-r(T-t)}F(t)$ we would short a forward contract and buy the stock; this would guarantee the risk-free profit. Similarly we cannot

† Note that $c(T-t)$, $p(T-t)$ and $\alpha(T-t)$ are functions of $(T-t)$, whereas $r(T-t)$ is simply a product.

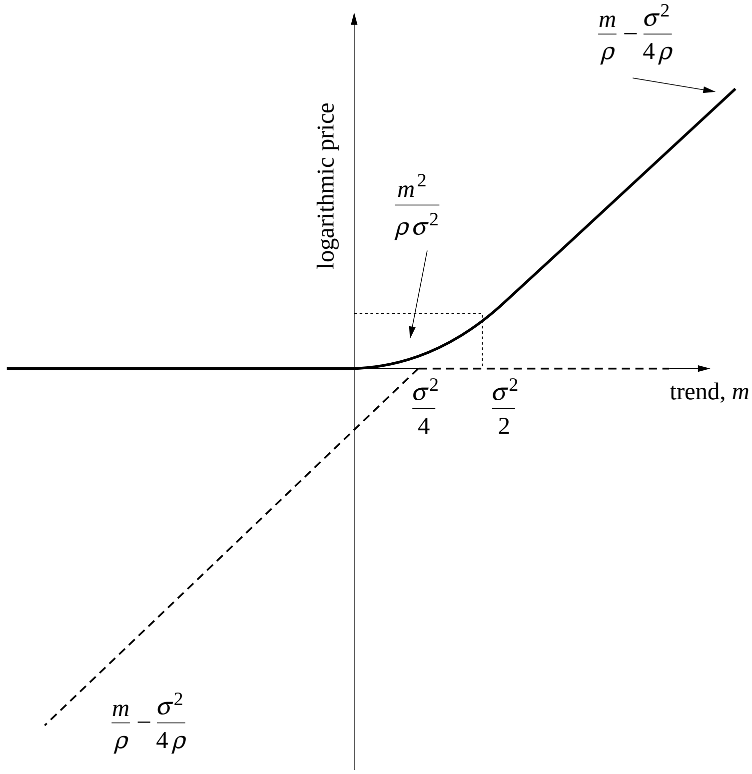

Figure 1. Bid–ask spread as a function of the market trend. The solid line represents $\log(S_p/S)$ and the dashed line represents $\log(S_s/S)$.

have $S_s(t) > \inf_{T\geqslant t} e^{-r(T-t)} F(t)$. These definitions impose a strict restriction on the possible values of the parameter, λ. From formula (3.6) it follows that $\lambda > \frac{1}{2}\sigma$ implies that S_p is infinite and $\lambda < \frac{1}{2}\sigma$ yields S_s as zero. Clearly, this is not the case in practice; therefore, we must have $\lambda = \frac{1}{2}\sigma$. This choice for λ also gives the Black–Scholes formula for a price of an option in the limit $\rho \to \infty$, see Black & Scholes (1973).

A general form of the purchase and sale prices is shown in figure 1, where the solid line represents $\log(S_p/S)$ and the dashed line denotes $\log(S_s/S)$. Note that $\log(S_p/S)$ is quadratic if $0 < m < \frac{1}{2}\sigma^2$ and linear if $m \geqslant \frac{1}{2}\sigma^2$.

Note that availability of the stock for $S(T)$ at maturity of the futures contract depends on the market trend at time T. In order to avoid having to consider forward purchase and sale prices (which makes modelling very complicated) we allow for cash settlement in extreme cases.

In figure 2 the dynamics of the change in the exchange rate between the Ukrainian currency (UAK) and the American dollar (USD) during the period from 15 to 23 April 1996 is shown. The low and high columns correspond to the purchase and sale prices of one dollar. The decline in the currency market has been caused by the supply of the USD that exceeded demand by \$79.5 m over the week on the Ukrainian currency exchange, according to the information supplied by the consulting firm ‘Intermarket’.

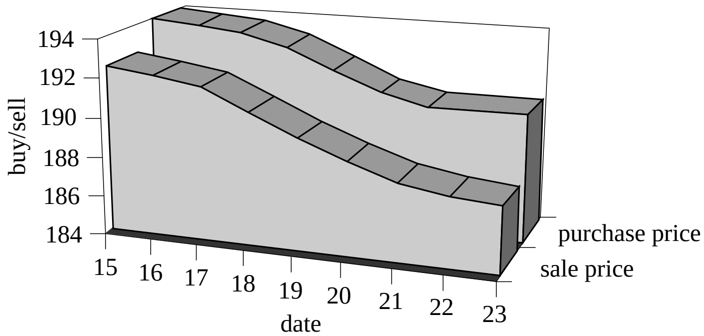

Figure 2. The purchase and sale prices of the US dollar in an illiquid (Ukranian) market.

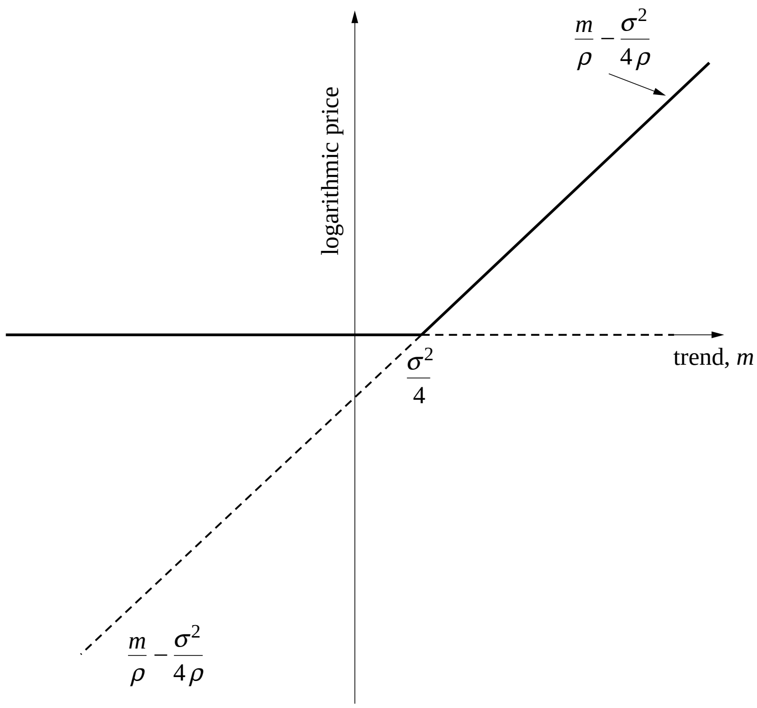

Figure 3. Simplified (logarithmic) sale and purchase prices.

One can see how the difference between the average bid and ask prices increases as the market falls. In general this kind of behaviour is common for illiquid markets.

4. Parameter estimation

In general, finding σ and ρ from the realized bid–ask spread is a difficult problem. However, a simplification is possible if the logarithm of the purchase price is approximated by a piecewise linear function as shown in figure 3.

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From the definition of the purchase and sale prices it is easy to see that

$$\frac{m_t}{\rho} - \frac{\sigma^2}{4\rho} = \operatorname{sgn}\left(\frac{m_t}{\rho} - \frac{\sigma^2}{4\rho}\right) \log\left(\frac{S_{\mathrm{p}}}{S_{\mathrm{s}}}\right). \tag{4.1}$$

In principle, one can observe the sign of the expression

$$\frac{m}{\rho} - \frac{\sigma^2}{4\rho}$$

following the dynamics of the market prices. Indeed, a change of sign corresponds to a change from a falling market to a rising market, or vice versa, when passing through the point of the tightest bid–ask spread, $m = \frac{1}{4}\sigma^2$.

The process followed by m_τ/ρ is ergodic since the correlation function

$$\tilde{k}(\tau, \sigma, \rho) = \frac{\sigma^2}{2\rho}\mathrm{e}^{-\rho|\tau|} \tag{4.2}$$

tends to zero as $\tau \to \infty$. For ergodic stationary processes, one realization of a long duration, T, is equivalent, in the sense of the volume of information, to a number of realizations of the same total duration. Therefore, the parameters of the process may be found as time averages, i.e.

$$0 = \mathcal{E}\left(\frac{m_t}{\rho}\right) \approx \frac{1}{T}\int_0^T \frac{m_t}{\rho}\,\mathrm{d}t, \tag{4.3}$$

$$\tilde{k}(\tau) = \mathcal{E}\left(\frac{m_t}{\rho}\frac{m_{t+\tau}}{\rho}\right) \approx \frac{1}{T-\tau}\int_0^{T-\tau} \frac{m_t}{\rho}\frac{m_{t+\tau}}{\rho}\,\mathrm{d}t. \tag{4.4}$$

On applying (4.3) we find that

$$-\frac{\sigma^2}{4\rho} \approx \frac{1}{T}\int_0^T \operatorname{sgn}\left(\frac{m_t}{\rho} - \frac{\sigma^2}{4\rho}\right) \log\left(\frac{S_{\mathrm{p}}(t)}{S_{\mathrm{s}}(t)}\right) \mathrm{d}t. \tag{4.5}$$

Now consider the division of the time-interval $[0, T]$ into n equal segments each of duration δt. The displacement, $-\sigma^2/(4\rho)$, in equation (4.1) is given by formula (4.5). Therefore, applying expression (4.4) for $\tau = i\delta t$, $i = 0, \dots, n$ we obtain empirical correlation, $\tilde{k}_i$. In order to get estimates of the parameters the least-squares method is employed and we minimize the function

$$\sum_{i=0}^{n}(\tilde{k}_i - \tilde{k}(i\delta t, \sigma, \rho))^2 \tag{4.6}$$

over $\sigma^2 \geqslant 0$ and $\rho \geqslant 0$.

Note that the role of relation (4.5) has been restricted to finding *driftless* measurements. In practice the bid–ask spread contains *other* components as well, and it is relatively difficult to separate them. Therefore, it would be unreasonable to treat (4.5) as a relation between σ and ρ. The minimization procedure (4.6) is based on finding correlation function (4.2), which best fits unbiased market data, and seems to be more appropriate in this case.

5. Hedging an option with the underlying asset

Although in our case risk cannot be completely eliminated when hedging an option only with the asset, we attempt to construct a strategy to reduce the risk efficiently. This can be particularly useful for hedging a tailor-made derivative when no other claims on the same underlying are available in the market.

As has been explained earlier it is hard to exploit market trends to obtain a riskless profit, since the bid–ask spread incorporates this effect already. Instead, risk-minimizing hedging strategy will be of interest in this work. Stochastic transaction costs incurred when hedging (the bid–ask spread) will be taken into account implicitly, since the problem becomes very complicated.

We consider the problem under the realistic assumption $\rho/\sigma^2 \gg 1$, i.e. the market is relatively liquid, and obtain an asymptotic expansion of the backward probability density function, $p = p(t, S, m, V)$ for the stock price S, trend m and wealth V, in powers of $\rho^{-1/2}$. In order to achieve the correct balance in the expansion we introduce $\tilde{m} = \rho^{-1/2} m$. Later it will become clear from the structure of the problem that this is the only consistent choice of the scalings.

In order to non-dimensionalize the system, the stock price, S, and the wealth, V, are rescaled with a unit of the currency. We should scale the time to the expiry and the market trend, m, with their typical value, σ^2. However, for the sake of simplicity of calculation we do not do this explicitly (as it does not change the answer in our case).

Let $y^* = y^*(t, S, \tilde{m}, V)$ denote the number of shares held at each moment of time. For a self-financing trading strategy the wealth, V, is driven by the process

$$\mathrm{d}V = r(V - Sy^*)\,\mathrm{d}t + y^*\,\mathrm{d}S = rV\,\mathrm{d}t + \rho^{1/2}\tilde{m}Sy^*\,\mathrm{d}t.$$

It will be verified below by direct calculation that the leading-order wealth satisfies the Black–Scholes equation if y^* admits the following representation:

$$y^*(t, S, \tilde{m}) = \Delta(t, S) - \Delta_S S \frac{\tilde{m}}{\rho^{1/2}} + \cdots, \tag{5.1}$$

where $\Delta(t, S)$ is the Black–Scholes delta of an option. This expansion is inferred from the intuitive argument, according to which risk is better reduced if the stock price is discounted by the current market trend, i.e. $y^* = \Delta(t, S\exp(-m/\rho))$. According to the Kolmogorov formula, the relation

$$p(t, S, \tilde{m}, V) = \mathcal{E}(f(S, \tilde{m}, V) \mid S(T) = S,\ \tilde{m}(T) = m,\ V(T) = V) \tag{5.2}$$

gives a solution to the Kolmogorov backward equation

$$\frac{\partial p}{\partial t} = rV\frac{\partial p}{\partial V} + \rho^{1/2}\tilde{m}S\Delta\frac{\partial p}{\partial V} + rS\frac{\partial p}{\partial S} + \rho^{1/2}\tilde{m}S\frac{\partial p}{\partial S} - \rho\tilde{m}\frac{\partial p}{\partial \tilde{m}} + \tfrac{1}{2}\sigma^2\rho\frac{\partial^2 p}{\partial \tilde{m}^2},$$

with the final condition $p(T, S, \tilde{m}, V) = f(S, \tilde{m}, V)$ for a sufficiently smooth function f.

Consider now the following expansion:

$$p = p_0 + \frac{1}{\rho^{1/2}}p_1 + \frac{1}{\rho}p_2 + \frac{1}{\rho^{3/2}}p_3 + \cdots.$$

The leading-order problem is

$$\tfrac{1}{2}\sigma^2\frac{\partial^2 p_0}{\partial\tilde{m}^2} - \tilde{m}\frac{\partial p_0}{\partial\tilde{m}} = 0,$$

which implies that

$$p_0 = p_0^0 + p_0^1 \int_0^{\tilde{m}} \mathrm{e}^{h^2/\sigma^2}\,\mathrm{d}h,$$

where p_0^0 and p_0^1 are functions of t, S, V but not $\tilde{m}$. The solvability condition for this problem yields $p_0^1 = 0$, in order to eliminate exponential growth at infinity.

At $O(\rho^{-1/2})$ we find that

$$\tilde{m}\frac{\partial p_1}{\partial\tilde{m}} - \tfrac{1}{2}\sigma^2\frac{\partial^2 p_1}{\partial\tilde{m}^2} = \tilde{m}S\frac{\partial p_0}{\partial S} + \tilde{m}S\Delta\frac{\partial p_0}{\partial V}.$$

By solving this equation and applying the solvability condition we find that

$$p_1 = p_1^0 + \tilde{m}S\left(\frac{\partial p_0}{\partial S} + \Delta\frac{\partial p_0}{\partial V}\right),$$

where p_1^0 is a function of t, S, V.

Finally we find that

$$-\tilde{m}^2\left\{S\frac{\partial p_0}{\partial S} + \Delta S\frac{\partial p_0}{\partial V} + S^2\frac{\partial^2 p_0}{\partial S^2} + 2\Delta S^2\frac{\partial^2 p_0}{\partial V\partial S} + (S\Delta)^2\frac{\partial^2 p_0}{\partial V^2}\right\} + \frac{\partial p_0}{\partial t} - rV\frac{\partial p_0}{\partial V} - rS\frac{\partial p_0}{\partial S} = \tfrac{1}{2}\sigma^2\frac{\partial^2 p_2}{\partial\tilde{m}^2} - \tilde{m}\frac{\partial p_2}{\partial\tilde{m}},$$

which yields

$$p_2 = \frac{2}{\sigma^2}L(p_0)\left\{\int_0^{\tilde{m}} \mathrm{e}^{x^2/\sigma^2}\,\mathrm{d}x \int_0^{\tilde{m}} \mathrm{e}^{-y^2/2\sigma^2}\,\mathrm{d}y - \int_0^{\tilde{m}}\int_0^{\tilde{m}} \mathrm{e}^{-x^2/2\sigma^2}\mathrm{e}^{\xi^2/2\sigma^2}\,\mathrm{d}\xi\,\mathrm{d}x\right\} + \cdots,$$

where

$$L(p_0) = \frac{\partial p_0}{\partial t} - (rV + \tfrac{1}{2}\sigma^2\Delta S)\frac{\partial p_0}{\partial V}(r + \tfrac{1}{2}\sigma^2)S\frac{\partial p_0}{\partial S} - \tfrac{1}{2}\sigma^2\left\{S^2\frac{\partial^2 p_0}{\partial S^2} + 2\Delta S^2\frac{\partial^2 p_0}{\partial V\partial S} + (S\Delta)^2\frac{\partial^2 p_0}{\partial V^2}\right\}.$$

On applying the solvability condition, p_0 is found to satisfy the equation

$$L(p_0) = 0. \tag{5.3}$$

Rather than applying formula (5.2) directly, one can notice that equation (5.3) implies that

$$\left.\begin{aligned} \mathrm{d}V &= rV\,\mathrm{d}t + \tfrac{1}{2}\sigma^2\Delta S\,\mathrm{d}t + \sigma\Delta S\,\mathrm{d}\beta, \\ \mathrm{d}S &= (r + \tfrac{1}{2}\sigma^2)S\,\mathrm{d}t + \sigma S\,\mathrm{d}\beta. \end{aligned}\right\} \tag{5.4}$$

Now we apply Itô's lemma to the left-hand side of equation (5.4) and choose $\Delta = \partial V/\partial S$. This yields the Black–Scholes equation for the leading order wealth:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$$

To see the importance of the order $\rho^{-1/2}$ term in representation (5.1), one can repeat the preceding analysis without that term, to find that (hedging in accordance with the pure Black–Scholes strategy) the wealth satisfies the equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Clearly, this equation cannot provide a satisfactory solution to the problem of the efficient risk management.

Representation (5.1) also suggests holding fewer shares than the Black–Scholes delta when the market is going up and more when it is falling. This might seem to be unreasonable at first sight but it is *optimal* in the sense of risk reduction in our problem, not to mention that buying (selling) extra shares in a rising (falling) market is *expensive* because of the large bid–ask spread.

6. Conclusions and discussions

In this paper we establish the relation between market trends, unavailability of the asset at its equilibrium price and the bid–ask spread. Having assumed that trends obey the Ornstein–Uhlenbeck process we find the tight bounds for the realized bid–ask spread to eliminate risk-free opportunities. We also consider hedging an option with another option as well as the underlying asset itself, in case no other contingent claims on the same asset are traded in the market. In the last case we modify the Black–Scholes strategy to obtain the best means for risk reduction.

Finally, we briefly outline the main results when a more sophisticated model for a stock-price behaviour, system (2.1)–(2.2), is employed. In the underdamped case, $4\omega^2 - \gamma^2 > 0$, the option pricing equation

$$\frac{\partial V}{\partial t} + S(r+m)\frac{\partial V}{\partial S} + \ell\frac{\partial V}{\partial m} + \tfrac{1}{2}\sigma^2\omega^2\frac{\partial^2 V}{\partial \ell^2} - (\sigma\omega\lambda + \gamma\ell + \omega^2 m)\frac{\partial V}{\partial \ell} - rV = 0$$

admits the following Green's function:

$$g(x,m,\ell,\tau) = \frac{\mathrm{e}^{-r\tau}}{\sqrt{2\pi\sigma^2\tilde{q}(\tau)}}\exp\left(-\frac{(r\tau - \sigma\lambda\tilde{p}(\tau) + x + m\tilde{a}(\tau) + \ell\tilde{b}(\tau))^2}{2\sigma^2\tilde{q}(\tau)}\right),$$

where

$$\tilde{p}(\tau) = \tau - \frac{1}{\beta^2+\omega_1^2}\left(\mathrm{e}^{-\beta\tau}\frac{\omega_1^2-\beta^2}{\omega_1}\sin(\omega_1\tau) + 2\beta(1-\mathrm{e}^{-\beta\tau})\cos(\omega_1\tau)\right),$$

$$\tilde{q}(\tau) = \int_0^\tau \left(1 - \mathrm{e}^{-\beta t}\frac{1}{\omega_1}(\beta\sin(\omega_1 t) + \omega_1\cos(\omega_1 t))\right)^2 \mathrm{d}t,$$

$$\tilde{a}(\tau) = \frac{1}{\omega^2}\left(2\beta - \mathrm{e}^{-\beta\tau}\left(\frac{\beta^2-\omega_1^2}{\omega_1}\sin(\omega_1\tau) - 2\beta\cos(\omega_1\tau)\right)\right),$$

$$\tilde{b}(\tau) = \frac{1}{\omega^2}\left(1 - \frac{\mathrm{e}^{-\beta\tau}}{\omega_1}(\beta\sin(\omega_1\tau) + \omega_1\cos(\omega_1\tau))\right),$$

$$\beta = \tfrac{1}{2}\gamma, \qquad \omega_1^2 = \omega^2 - \beta^2 > 0,$$

and x, g, τ have been introduced earlier.

By means similar to those detailed above, the forward price is found to be given by

$$F = S\exp(r(T-t) - \lambda\sigma\tilde{p}(T-t) + \tfrac{1}{2}\sigma^2\tilde{q}(\tau) + m\tilde{a}(T-t) + \ell\tilde{b}(T-t)),$$

and the purchase and sale prices are defined as the supremum and infimum of the discounted forward price, respectively. This again imposes the condition $\lambda = \frac{1}{2}\sigma$. The overdamped and critically damped cases can be dealt with similarly, although these cases are less likely to be of interest in practice.

When hedging with the underlying asset, the optimal risk-reducing strategy is to hold

$$y^* = \Delta(t,S) - \Delta_S S\frac{\ell}{\omega^2} + \cdots$$

shares for $\omega/\sigma^2 \gg 1$. This can be verified as in § 5.

Illiquid markets are also notorious for their crashes. The model presented here can be adapted to explain this sort of behaviour if we introduce a feed-back effect by making the damping coefficient γ a function of the trend m; for example, $\gamma = (1 - 3m^2)$. Financially this means that trends make investment or short selling more attractive and hence promote information propagation between trades. The model predicts, under these circumstances, catastrophic collapses and bull runs corresponding to the relaxation oscillations of Van der Pol's type. 'A tremendous crash in the stock market is a relaxation oscillation—one cycle. A relaxation oscillation is characterized by long intervals of quiescence followed by a very sudden, sometimes catastrophic change over a very short period of time' (see Lindsey 1972, p. 239). For a time analysis of the S&P 500 index during the October 1987 market crash see Bouchaud & Sornette (1996). They confirm that the index behaviour was similar to relaxation oscillations.

The detailed investigation of possible feed-back effects of trading, as well as optimal hedging under this assumption is a topic for further research.

Appendix A.

Market trends are not the only reason for the unavailability of the underlying. In some markets an inability to trade can be caused purely by technical reasons. In such cases, in general, the times during which the asset is available for purchase appear irregularly and unpredictably. We shall assume that the intervals of time during which the asset is available are separated by time intervals drawn from independent identical distributions. If these intervals are also exponentially distributed, $F(x) = 1 - \exp(\kappa x)$, where $F(x)$ is the cumulative density function and $\kappa > 0$, then the moments of the stock arrival obey Poisson's law. For many practical purposes this is a sensible assumption, as in most cases it allows us to obtain compact results that quantify the effects of the random availability well enough.

Let the stock price, S, be driven by the following discrete stochastic equation:

$$\delta S = \mu S\delta t + \sigma S\phi\sqrt{\delta t},$$

where $\mu > 0$ and $\sigma > 0$ are the growth rate and volatility, respectively; δt is a small and random time increment, distributed in accordance with the law $F(x)$ and ϕ is a normally distributed random variable with zero mean and unit variance, independent of δt. The increments δt are also independent of each other as well as the current stock price S.

It is easy to see that

$$\mathcal{E}(\delta t) = \int_0^\infty x\, \mathrm{d}F(x) = \frac{1}{\kappa}, \qquad \mathcal{E}(\delta t^2) = \int_0^\infty x^2\, \mathrm{d}F(x) = \frac{2}{\kappa^2}.$$

The parameter κ can be interpreted as the expected number of rehedges per unit time. Hence, an infinite value of κ implies the ability to hedge continuously in time and should bring us back to the classical Black–Scholes world.

As in the previous sections, to find the value of an option as well as the risk-minimizing hedging strategy, a portfolio containing one option and Δ number of shares will be set up. The quantity Δ is chosen at the first moment when the underlying becomes available to hedge an option with and remains fixed during the next random time-step δt (during which the asset is unavailable).

Although the portfolio $\Pi = V - \Delta S$, where $V = V(S,t)$ is the current price of an option, cannot possibly be made risk free in the present situation, its increment $\delta\Pi$ is set equal to $r\Pi\delta t$, and the residual risk will be taken into account later. An estimate of the risk will be based upon the average number of rehedges, which are for Poisson's law equal to κT, where $T > 0$ is the lifespan of an option.

By employing stochastic Taylor series the increment of the portfolio Π can be expressed as

$$\delta\Pi = \delta V - \Delta\delta S = \frac{\partial V}{\partial t}\delta t + \tfrac{1}{2}\sigma^2 S^2\phi^2\frac{\partial^2 V}{\partial S^2}\delta t + \left(\frac{\partial V}{\partial S} - \Delta\right)\delta S.$$

It can be easily verified that the risk of the hedged portfolio is given by

$$\begin{aligned}\operatorname{var}(\delta\Pi)(\Delta) &= \mathcal{E}(\delta\Pi^2) - \mathcal{E}(\delta\Pi)^2\\ &= \frac{1}{\lambda^2}\left[\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \left(\frac{\partial V}{\partial S} - \Delta\right)\mu S\right]^2 + \frac{1}{\lambda}\sigma^2 S^2\left(\frac{\partial V}{\partial S} - \Delta\right)^2.\end{aligned} \tag{A 1}$$

In order to minimize the risk exposure, the derivative of the hedged portfolio with respect to Δ is set equal to zero, i.e.

$$\frac{\partial \operatorname{var}(\delta\Pi)(\Delta)}{\partial\Delta} = 2\frac{\mu S}{\kappa^2}\left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right) + 2\left(\Delta - \frac{\partial V}{\partial S}\right)\left(\frac{\mu^2}{\kappa^2} + \frac{\sigma^2}{\kappa}\right)S^2 = 0,$$

and hence

$$\Delta = \frac{\partial V}{\partial S} + \left(\frac{1}{S}\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)\Big/\left(\mu + \kappa\frac{\sigma^2}{\mu}\right). \tag{A 2}$$

It is easy to see that with this choice of Δ, expression (A 1) attains its global minimum since

$$\frac{\partial^2 \operatorname{var}(\delta\Pi)}{\partial\Delta^2} = \frac{2\mu^2 S^2}{\lambda^2} + \frac{2\Delta\sigma^2 S^2}{\lambda} > 0$$

and when $\Delta \to \pm\infty$ expression (A 1) tends to $+\infty$.

By substituting expression (A 2) into the pricing formula,

$$\mathcal{E}(\delta\Pi) = r\mathcal{E}(\Pi\delta t),$$

the modified Black–Scholes equation,

$$\frac{\partial V}{\partial t} + \bar{r}S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\bar{r}V = 0, \tag{A 3}$$

is obtained, where $\bar{r}$ is defined to be

$$\bar{r} = r\frac{\mu + \kappa\sigma^2/\mu}{r + \kappa\sigma^2/\mu}.$$

In our case the argument about the construction of a completely risk-free position is impossible and, hence, the risk-neutral world is irrelevant. This is why the modified interest rate $\bar{r}$ will always be higher than the actual interest rate r (assuming that $\mu > r$). The limit $\kappa \to \infty$ gives us back the Black–Scholes equation with the actual interest rate r.

Taking into account (A 3), expression (A 2) for the optimal number of shares to hold admits a simplification:

$$\Delta = \frac{\partial V}{\partial S}\frac{\kappa\sigma^2/\mu}{r + \kappa\sigma^2/\mu} + \frac{V}{S}\frac{r}{r + \kappa\sigma^2/\mu}.$$

The risk due to hedge imperfections can be estimated using the result obtained by Bouchaud & Sornette (1994). According to them the residual risk (variance), J, is given by

$$J = \frac{\sigma^2}{8\kappa} + O\left(\frac{1}{\kappa^2}\right).$$

The residual risk can be priced in accordance with the investor's attitude towards uncertainty.

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